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LETTER TO THE EDITOR

Universality of eigenvector statistics of kicked tops of different symmetries

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Abstract. We show that the eigenvectors of the Floquet operators of periodically kicked tops with orthogonal, unitary and symplectic canonical transformations conform to the predictions of the respective circular ensembles of random matrices.

The Gaussian and circular ensembles of random matrices [1] have proven successful in their predictions for the eigenvalue statistics of quantum Hamiltonians H and Floquet operators F which classically describe fully developed chaos. The level spacing distribution and the spectral stiffness, as well as other characteristics of the spectrum of H or F, are generically indistinguishable from the corresponding properties of matrices randomly drawn from the appropriate ensemble [2-6].

Much less is known about the reliability of random matrix theory with respect to the statistics of eigenvectors of H or F of dynamical systems [7]. We have therefore subjected the eigenvectors of the Floquet operator of periodically kicked tops [3-6, 8, 9] to statistical analysis. As will be shown below the eigenvector statistics agree nicely with that derived from Dyson's circular ensembles of random matrices. Especially, tops pertaining to each of the three known universality classes have been investigated and found to clearly reveal the rather different behaviour of matrices from the orthogonal, unitary or symplectic circular ensemble.

Considering first the circular unitary ensemble (CUE) of $N\times N$ unitary matrices F we are facing unit-norm eigenvectors with N in general complex components $c_n = c'_n + ic''_n$. For a fixed matrix F every eigenvector can be unitarily transformed into an arbitrarily prescribed unit vector. The only characteristic of eigenvectors invariant under the unitary canonical transformations§ is the norm itself. The joint probability density for the N complex components of the eigenvectors must therefore be [10]

$$P_{\mathrm{CUE}}(\{c_n\}) = \text{constant}\times\delta\left(1-\sum_{n=1}^{N}|c_n|^2\right) \qquad (1)$$

the constant being fixed by normalisation. Evidently, $P_{\mathrm{CUE}}(\{c_n\})$ is non-zero only on the surface of a d-dimensional unit sphere with $d=2N$. A convenient quantity to compare with data pertaining to the Floquet operator of a dynamical system is the

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§ We refer to transformations as canonical when they preserve the eigenvalues of F, the unitarity $F^+F=1$, and the transformation properties, if any, under time reversal. For the CUE, COE and CSE the groups of canonical transformations are, respectively, $\mathrm{U}(N)$, $\mathrm{O}(N)$, and $\mathrm{Sp}(N)$.

reduced density of, say, $|c_1|^2$, i.e. of the first-basis-state population in an eigenstate of F,

$$P_{\mathrm{CUE}}(A)=\int \mathrm{d}^2c_1\ldots \mathrm{d}^2c_N\,\delta(A-|c_1|^2)P_{\mathrm{CUE}}(\{c\}). \tag{2}$$

For the circular orthogonal ensemble of unitary $N\times N$ matrices the eigenvectors can be taken as real. The probability density $P_{\mathrm{COE}}(\{c\})$ of the circular orthogonal ensemble (COE) is thus concentrated on the surface of a unit sphere of dimension $d=N$,

$$P_{\mathrm{COE}}(\{c\})=\text{constant}\times\delta\left(1-\sum_{n=1}^{N}c_n^2\right) \tag{3}$$

and the reduced density for any particular basis state being populated is

$$P_{\mathrm{COE}}(A)=\int \mathrm{d}c_1\ldots \mathrm{d}c_N\,\delta(A-c_1^2)P_{\mathrm{COE}}(\{c\}). \tag{4}$$

Finally, the matrices from the circular symplectic ensemble (CSE) again have unit-norm eigenvectors with N in general complex components, N being an even number. The density $P_{\mathrm{CSE}}(\{c\})$ therefore is identical in appearance with $P_{\mathrm{CUE}}(\{c\})$. However, the natural reduced density to compare with the data for the Floquet operator of a dynamical system now looks different from its COE and CUE analogues. We must recall that in the symplectic case there is an antiunitary time-reversal operator T squaring to minus unity which yields a time-reversal 'covariance' of F

$$TFT^{-1}=F^{+} \qquad T^2=-1. \tag{5}$$

A natural basis to work with has the structure

$$|1\rangle,\ T|1\rangle,\ |2\rangle,\ T|2\rangle,\ldots,|N/2\rangle,\ T|N/2\rangle. \tag{6}$$

The two eigenvectors of F pertaining to one given eigenvalue can then be written as

$$\begin{aligned}|e_1\rangle&=c_1|1\rangle+\tilde{c}_1T|1\rangle+c_2|2\rangle+\tilde{c}_2T|2\rangle+\ldots\\ T|e_1\rangle&=-\tilde{c}_1{}^*|1\rangle+c_1^*T|1\rangle-\tilde{c}_2{}^*|2\rangle+c_2^*T|2\rangle+\ldots.\end{aligned} \tag{7}$$

However, any linear combination of $|e_1\rangle$ and $T|e_1\rangle$ together with an orthogonal one,

$$\alpha|e_1\rangle+\beta T|e_1\rangle \qquad -\beta|e_1\rangle+\alpha T|e_1\rangle \tag{8}$$

with $|\alpha|^2+|\beta|^2=1$ can equally well serve as a pair of eigenvectors pertaining to the eigenvalue in consideration. A diagonalisation routine starting with the matrix F in the representation (6) will in general not yield the two eigenvectors in the form (7) rather than (8). The probability of having the state $|1\rangle$ populated is thus as much a property of the diagonalisation routine as one of the eigenvectors of F. However, the occupation probability $|c_1|^2+|\tilde{c}_1|^2$ of the two-dimensional subspace spanned by $|1\rangle$ and $T|1\rangle$ is invariant under the transformation from (7) to (8). We are thus led to comparing the reduced density

$$P_{\mathrm{CSE}}(A)=\int \mathrm{d}^2c_1\,\mathrm{d}^2\tilde{c}_1\ldots \mathrm{d}^2c_{N/2}\,\mathrm{d}^2\tilde{c}_{N/2}\delta(A-|c_1|^2-|\tilde{c}_1|^2)P_{\mathrm{CSE}}(\{c,\tilde{c}\}) \tag{9}$$

with the corresponding distribution obtained by diagonalising F where

$$P_{\mathrm{CSE}}(\{c,\tilde{c}\})=\text{constant}\times\delta\left(1-\sum_{n=1}^{N/2}(|c_n|^2+|\tilde{c}_n|^2)\right). \tag{10}$$

For all three circular matrix ensembles the joint distribution of the components of eigenvectors are uniformly concentrated on d-dimensional unit spheres with $d = N$ in the orthogonal case and $d = 2N$ for the unitary and symplectic ensembles. In a unified notation using real variables, $x_1, \ldots, x_d$ the properly normalised joint distribution is

$$p^{(d)}(\{x\}) = \pi^{-d/2}\Gamma(d/2)\delta\left(1 - \sum_{n=1}^{d} x_n^2\right). \tag{11}$$

By integrating out $d - l$ of the variables x we obtain the reduced distribution

$$P^{(d,l)}(x_1, \ldots, x_l) = \pi^{-l/2}\frac{\Gamma(d/2)}{\Gamma[(d-l)/2]}\left(1 - \sum_{n=1}^{l} x_n^2\right)^{(d-l-2)/2}. \tag{12}$$

The $(d-l)$-fold integral over $P^{(d)}(\{x\})$ is most conveniently carried out by using a Fourier integral representation for the delta function in (11). Different choices for d and l now give the reduced densities $P_{\mathrm{COE}}(A)$, $P_{\mathrm{CUE}}(A)$ and $P_{\mathrm{CSE}}(A)$ defined above.

In the orthogonal case we must take $d = N$, $l = 1$ and find

$$P_{\mathrm{COE}}(A) = \pi^{-1/2}\Gamma(N/2)/\Gamma((N-1)/2)A^{-1/2}(1-A)^{(N-3)/2}. \tag{13}$$

The analogous result for the unitary case is obtained with $d = 2N$, $l = 2$ as

$$P_{\mathrm{CUE}}(A) = (N-1)(1-A)^{N-2} \tag{14}$$

while the symplectic case requires $d = 2N$, $l = 4$ and leads to

$$P_{\mathrm{CSE}}(A) = (N-1)(N-2)A(1-A)^{N-3}. \tag{15}$$

Interestingly, the three functions (13)-(15) are quite different in their A dependence. It is well to remark that nowhere in constructing the $P(\{c\})$ and the $P(A)$ have we made use of the unitarity of the matrices F. In fact, the results (13)-(15) hold for the Gaussian ensemble of Hermitian matrices as well.

Periodically kicked tops have proved an important testing ground for universality of quantum fluctuations under conditions of classical chaos [3-6, 8, 9]. The relevant dynamical variables are the three components of an angular momentum operator $\boldsymbol{J}$ which obey the commutation relations $[J_i, J_j] = i\varepsilon_{ijk}J_k$. Since all tops have the squared angular momentum conserved there is a good quantum number j defined by $\boldsymbol{J}^2 = j(j+1)$. All integer and half-integer numbers are allowed for j. While classical behaviour arises for $j \to \infty$ we are interested in the semiclassical range $1 \ll j < \infty$. For fixed j the dimension of the Hilbert space is $N = 2j + 1$. Periodic kicking is described by a Hamiltonian of the form

$$H = H_0 + V\sum_{n=-\infty}^{+\infty}\delta(t-n) \tag{16}$$

if the kicking period is set equal to unity. The operators H_0 and V are polynomials in $\boldsymbol{J}$. Restricting ourselves to a period-to-period stroboscopic description we have to deal with the unitary Floquet operator

$$F = \mathrm{e}^{-\mathrm{i}V}\,\mathrm{e}^{-\mathrm{i}H_0}. \tag{17}$$

Different choices for H_0 and V now allow us to construct Floquet operators from different universality classes. The simplest case to realise is the one with $\mathrm{O}(N)$ as the group of canonical transformations. By choosing [4, 9]

$$V = \lambda J_z^2/2j \qquad H_0 = pJ_y \tag{18}$$

we have a Floquet operator with the covariance

$$TFT^{-1} = F^{+} \tag{19}$$

under the antiunitary time-reversal operation

$$T = \mathrm{e}^{\mathrm{i}H_0}\,\mathrm{e}^{\mathrm{i}\pi J_x}\,T_0$$

where T_0 is the conventional time reversal, $T_0 J T_0^{-1} = -J$. The statistics of the eigenphases of F must therefore be expected and have indeed been shown to be that of the COE [4]. Curve O in figure 1 presents the analogous evidence for the eigenvectors, obtained by diagonalising F for $j = 100$, $\lambda = 6.0$, $p = 1.7$. Due to the geometric invariance $[R_y, F] = 0$, with R_y a rotation by π around the y axis, the matrix F breaks into two blocks, one with dimension j and the other with dimension $j+1$. The eigenvector data in figure 1 pertain to the j-dimensional case.

In order to construct a dynamics with symplectic canonical transformations we must again secure a time-reversal covariance of the form (19) but take a half-integer value of j in order to have $T^2 = -1$. Moreover, since there must be no geometric symmetry the operators H_0 and V cannot be quite as simple as (18). The presumably simplest case has [6]

$$\begin{aligned} H_0 &= pJ_z^2/j \\ V &= k[J_z^2/j + k'(J_xJ_z + J_zJ_x)/j + k''(J_xJ_y + J_yJ_x)/j] \end{aligned} \tag{20}$$

and $T = \mathrm{e}^{\mathrm{i}H_0}T_0$. Quartic level repulsion as characteristic for the CSE has been demonstrated in [6]. Curve S of figure 1 displays the quantitative reliability of the CSE prediction for the eigenvector statistics. The diagonalisation of F was carried out for $j = 49.5$, $p = 2.5$, $k = 2.5$, $k' = 2$, $k'' = 3$.

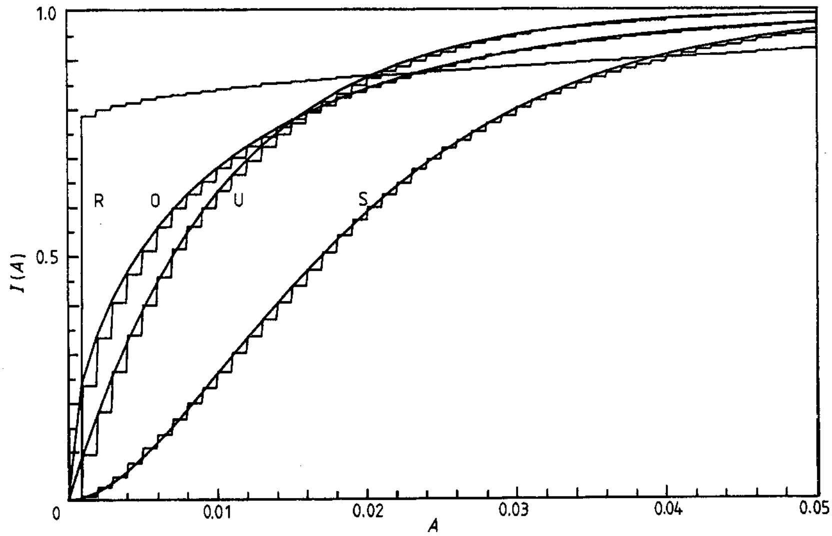

Figure 1. Integral $I(A) = \int_0^A \mathrm{d}a\, P(a)$ of the reduced probability P plotted against appropriate amplitude A for the four models of kicked tops: R, nearly integrable case with orthogonal symmetry, $A = c_1^2$; O, non-integrable case with orthogonal symmetry, $A = c_1^2$; U, non-integrable case with unitary symmetry, $A = |c_1|^2$; S, non-integrable case with symplectic symmetry, $A = |c_1|^2 + |\tilde{c}_1|^2$. The smooth lines correspond to the theoretical predictions (13)–(15) of COE, CUE and CSE whereas the histograms give the numerical diagonalisations. In all cases the dimensionality of matrices was equal to 100.

The simplest dynamics without a time-reversal symmetry that we have found [4] involves a Floquet operator consisting of three unitary factors

$$F = \exp(-\mathrm{i}k'J_x^2/2j)\exp(-\mathrm{i}kJ_z^2/2j)\exp(-\mathrm{i}pJ_y). \tag{21}$$

This differs from (17) and (18) by an additional non-linear kick around the x axis. For $k' = k$ and $p \neq \pi/2$ there is no antiunitary symmetry as was evidenced by quadratic level repulsion according to the CUE in [4]. Curve U in figure 1 shows that the eigenvectors also conform to the CUE prediction. The calculation was done for $j = 100$, $k = 6.0$, $k' = 0.5$.

Finally, curve R of figure 1 refers again to the dynamics (17) and (18) but with $k = 1$ to make the classical motion nearly integrable. The level spacings have Poissonian statistics in that case [6]. The eigenvector distribution shows that the eigenvectors have their supports on a very small number of basis states, just as one must expect for random matrices with a tendency to level clustering.

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